

\mathcal{D} -modules and complex foliations

Hamidou Dathe

June 30, 2016

Abstract

Consider a complex analytic manifold X and a coherent Lie subalgebra \mathcal{I} of the Lie algebra of complex vector fields on X . By using a natural \mathcal{D}_X -module $\mathcal{M}_{\mathcal{I}}$ naturally associated to \mathcal{I} and the ring (in the derived sense) $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_{\mathcal{I}}, \mathcal{M}_{\mathcal{I}})$, we associate integers which measure the irregularity of the foliation associated with \mathcal{I} .

1 Introduction

The idea of using \mathcal{D} -module theory in the study of foliations is very natural and appeared in particular in [8, 2]. More precisely, consider a complex manifold X and a coherent Lie subalgebra \mathcal{I} of the sheaf Θ_X of tangent vectors. To this ideal is naturally associated the coherent left \mathcal{D} -module $\mathcal{M}_{\mathcal{I}} = \mathcal{D}_X / \mathcal{D}_X \cdot \mathcal{I}$ already considered in [8]. The new idea of this paper is to consider the ring (in the derived sense) $\mathcal{D}_{\mathcal{I}} := R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_{\mathcal{I}}, \mathcal{M}_{\mathcal{I}})$ and $\mathcal{D}_{\mathcal{I}}^0 := H^0(\mathcal{D}_{\mathcal{I}})$. We denote by $\mathcal{D}\text{-irr}(\mathcal{I})$ the increasing sequence consisting of the integers k such that $H^k(\mathcal{D}_{\mathcal{I}}) \neq 0$ and call it the \mathcal{D} -irregularity of the foliation. These integers $k \in \mathbb{N}$ for which the cohomology of this ring is not 0 give invariant which measure, in some sense, the irregularity of the foliation. We call the biggest of these integers the \mathcal{D} -irregularity of the foliation. With some hypotheses, already considered in [8], we compute the \mathcal{D} -irregularity and calculate $\mathcal{D}\text{-irr}(\mathcal{I})$ for some examples of foliation. We also give geometrically interpretation of $\mathcal{D}\text{-irr}(\mathcal{I})$.

Acknowledgments

The author warmly thanks Pierre Schapira for proposing him this subject and for his advices during the preparation of the manuscript.

2 Foliations

In all this paper, X denotes a complex manifold of complex dimension d_X . One denotes by \mathcal{O}_X the structure sheaf, by Θ_X the sheaf of holomorphic vector fields, by Ω_X^1 the sheaf of holomorphic 1-forms and by \mathcal{D}_X the sheaf of holomorphic differential operators.

Vector fields and 1-forms

Consider first a locally free \mathcal{O}_X -module of finite rank \mathcal{L} and set

$$\mathcal{L}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X).$$

One denotes by $\langle \cdot, \cdot \rangle$ the pairing $(\mathcal{L}, \mathcal{L}^*) \rightarrow \mathcal{O}_X$. One uses the same notations when interwinning \mathcal{L} and \mathcal{L}^* .

Let \mathcal{I} be an \mathcal{O}_X -submodule of \mathcal{L} . Recall that \mathcal{I} is coherent if and only if it is locally finitely generated, that is, if there exists a locally free \mathcal{O}_X -module of finite rank \mathcal{K} and an \mathcal{O}_X -linear map $\psi: \mathcal{K} \rightarrow \mathcal{L}$ such that $\mathcal{I} = \text{Im } \psi$.

One defines the orthogonal \mathcal{I}^\perp in \mathcal{L}^* to \mathcal{I} by

$$\omega \in \mathcal{I}^\perp \Leftrightarrow \langle \omega, v \rangle = 0 \text{ for all } v \in \mathcal{I}.$$

More precisely, \mathcal{I}^\perp is the sheaf associated with the presheaf $U \mapsto \mathcal{I}(U)^\perp$.

Lemma 2.1. *Let \mathcal{I} be a coherent \mathcal{O}_X -submodule of \mathcal{L} . Then \mathcal{I}^\perp is coherent and $\mathcal{I} \xrightarrow{\sim} \mathcal{I}^{\perp, \perp}$.*

Proof. Consider $\psi: \mathcal{K} \rightarrow \mathcal{L}$ as above such that $\mathcal{I} = \text{Im } \psi$ and denote by $\psi^*: \mathcal{L}^* \rightarrow \mathcal{K}^*$ the dual map. Then

$$\begin{aligned} (\text{Im } \psi)^\perp &\simeq \text{Ker } \psi^*, \\ \text{Ker } (\psi^*)^\perp &\simeq \text{Im } \psi, \end{aligned}$$

and the result follows since $\text{Mod}_{\text{coh}}(\mathcal{O}_X)$ is abelian.

Q.E.D.

In the sequel, we shall apply Lemma 2.1 to the case where $\mathcal{L} = \Theta_X$ and thus $\mathcal{L}^* = \Omega_X^1$. Note that the duality is given by

$$\langle \sum_i a_i(x) df_i, v \rangle = \sum_i a_i(x) v(f_i), \quad v \in \Theta_X.$$

The next result is well-known.

Proposition 2.2. *Let \mathcal{J} be a coherent submodule of Θ_X . Then \mathcal{J} is a Lie subalgebra of Θ_X if and only if the coherent submodule \mathcal{J}^\perp of Ω_X^1 satisfies $d\mathcal{J}^\perp \subset \mathcal{J}^\perp \wedge \mathcal{J}^\perp$.*

Proof. For ω a section of Ω_X^1 and v_1, v_2 two sections of Θ_X , recall the formula

$$d\omega(v_1, v_2) = v_1 \cdot \langle \omega, v_2 \rangle - v_2 \cdot \langle \omega, v_1 \rangle - \langle \omega, [v_1, v_2] \rangle.$$

Choosing ω in \mathcal{J}^\perp , we find that $d\omega(v_1, v_2) = 0$ if and only if $\langle \omega, [v_1, v_2] \rangle = 0$. Hence

- (i) if \mathcal{J} is a Lie subalgebra, then $d\omega \in \mathcal{J}^\perp \wedge \mathcal{J}^\perp$,
and conversely:
- (ii) if $d\mathcal{J}^\perp \subset \mathcal{J}^\perp \wedge \mathcal{J}^\perp$, then $d\omega(v_1, v_2) = 0$ for any ω in \mathcal{J}^\perp , hence $[v_1, v_2]$ belongs to $\mathcal{J}^{\perp, \perp} \simeq \mathcal{J}$. Q.E.D.

Example 2.3. Consider a complex Poisson manifold X , that is, a complex manifold X endowed with a bracket $\{\cdot, \cdot\}: \mathcal{O}_X \times \mathcal{O}_X \rightarrow \mathcal{O}_X$ satisfying the Jacobi identities. One defines [2] the Lie sub-algebra \mathcal{J} of Θ_X as the ideal generated by the derivations $\{f, \cdot\}; f \in \mathcal{O}_X$.

Foliations

We consider now an \mathcal{O}_X -submodule \mathcal{J} of Θ_X and we assume that

$$(2.1) \begin{cases} \text{(i) } \mathcal{J} \text{ is } \mathcal{O}_X\text{-coherent,} \\ \text{(ii) } \mathcal{J} \text{ is a Lie subalgebra of } \Theta_X. \end{cases}$$

We call \mathcal{J} a singular foliation of X .

For $x \in X$, we denote by $\mathcal{J}(x)$ the subspace of $T_x X = \Theta_X(x)$ generated by the germs of sections of \mathcal{J} at x .

Definition 2.4. One sets

$$\begin{cases} \text{rk}(\mathcal{J}) = \sup_{x \in X} \dim \mathcal{J}(x), \\ \text{cork}(\mathcal{J}) = \inf_{x \in X} \dim \mathcal{J}(x), \\ \text{irr}(\mathcal{J}) = \text{rk}(\mathcal{J}) - \text{cork}(\mathcal{J}). \end{cases}$$

One says that $\text{rk}(\mathcal{J})$ is the rank of \mathcal{J} , $\text{cork}(\mathcal{J})$ the corank of \mathcal{J} and $\text{irr}(\mathcal{J})$ the irregularity of \mathcal{J} .

We set

$$X_j = \{x \in X; \dim \mathcal{J}(x) = \text{rk}(\mathcal{J}) - j\}.$$

Hence, the X'_j s are locally closed complex analytic submanifolds and one gets a stratification

$$X = \bigsqcup_{j=0}^{\text{irr}(\mathcal{J})} X_j.$$

Note that X_0 is an open dense subset of X , \mathcal{J} is locally free on X_0 and $X \setminus X_0$ is a closed complex analytic subset of X of codimension at least 1.

One says that the foliation is regular if $\text{irr}(\mathcal{J}) = 0$, that is, if $X_0 = X$.

Definition 2.5. Let $\Sigma \subset X$ be an embedded submanifold of X . One says that Σ is a leaf of \mathcal{J} if for any $x \in \Sigma$, $T_x \Sigma = \mathcal{J}(x)$.

Remark 2.6. (i) The Frobenius theorem asserts that each X_j admits a foliation by complex leaves of dimension $\text{rk}(\mathcal{J}) - j$.

(ii) There is a theorem by Nagano [6] which asserts that X is a unique disjoint union of connected leaves. This result is false in the C^∞ -setting as shown by the following example, due to Nagano. Let $X = \mathbb{R}^2$ endowed with coordinates (x, y) and consider the two vector fields on \mathbb{R}^2 given by $\partial_x, f(x)\partial_y$ and the left ideal \mathcal{J} they generate. Assume that $f(x) \neq 0$ for $x \neq 0$ and f has a zero of infinite order at $x = 0$. Then $\mathbb{R}^2 = \{x \neq 0\} \cup \{x = 0\}$ and the set $\{x = 0\}$ is not a leaf.

(iii) The Camacho-Sad theorem asserts that in dimension 2, each point of X belongs to the closure of a leaf of X_0 .

3 Links with \mathcal{D} -modules

The idea of associating a \mathcal{D} -module to a foliation is not new. See in particular [8] and for a systematic approach in the algebraic case, see [2].

We refer to Kashiwara [3] for \mathcal{D} -module theory.

Let \mathcal{J} satisfying (2.1). We denote by $\widetilde{\mathcal{J}}$ the left ideal of \mathcal{D}_X generated by \mathcal{J} , that is, $\widetilde{\mathcal{J}} = \mathcal{D}_X \cdot \mathcal{J}$. We denote by $\mathcal{M}_{\mathcal{J}} := \mathcal{D}_X / \widetilde{\mathcal{J}}$ the associated \mathcal{D}_X -module. Since \mathcal{J} is \mathcal{O}_X -coherent, the ideal $\widetilde{\mathcal{J}}$ is locally finitely generated,

hence coherent in \mathcal{D}_X and therefore $\mathcal{M}_{\mathcal{J}}$ is coherent. Such a module has already been considered and studied in [2].

Recall that to a coherent \mathcal{D}_X -module \mathcal{M} , one associates its characteristic variety $\text{char}(\mathcal{M})$, a closed co-isotropic \mathbb{C}^* -conic analytic subset of the cotangent bundle T^*X . Recall a definition of [3, p.19]

Definition 3.1. Let $B := \{P_1, \dots, P_r\}$ be a system of generators of a left ideal \mathcal{J} of \mathcal{D}_X . One says that B is an involutive system of generators of \mathcal{J} if

$\text{gr}B := \{\sigma(P_1), \dots, \sigma(P_r)\}$ is a system of generators of the graded ideal $\text{gr}\mathcal{J}$.

For such an involutive system of generators, we evidently have

$$\text{char}(\mathcal{D}_X/\mathcal{J}) = \{(x; \xi) \in T^*X; \sigma(P_j)(x; \xi) = 0, \text{ for all } 1 \leq j \leq r\}.$$

We shall consider the hypotheses

$$(3.1) \begin{cases} \text{there locally exist } v_1, \dots, v_r \in \mathcal{J} \text{ such that the family} \\ (v_1, \dots, v_r) \text{ generates } \mathcal{J}, [v_i, v_j] = 0 \text{ for } 1 \leq i, j \leq r \text{ and the} \\ \text{sequence of symbols } \{\sigma(v_1), \dots, \sigma(v_r)\} \text{ is a regular sequence} \\ \text{in } \mathcal{O}_{T^*X}, \end{cases}$$

and

$$(3.2) \begin{cases} \text{there locally exist } v_1, \dots, v_r \in \mathcal{J} \text{ such that the family} \\ (v_1, \dots, v_r) \text{ generates } \mathcal{J}, [v_i, v_j] = 0 \text{ for } 1 \leq i, j \leq r \text{ and} \\ \text{the sequence } \{v_1, \dots, v_r\} \text{ is a regular sequence in } \mathcal{D}_X. \end{cases}$$

Note that the hypothesis (3.1) has already been considered and studied in [8].

In order to compare hypotheses (3.1) and (3.2), recall some results from [3]. Consider a complex of filtered \mathcal{D}_X -modules

$$(3.3) \quad \mathcal{M}_1 \rightarrow \mathcal{M}_2 \rightarrow \mathcal{M}_3$$

and the associated complex of graded modules

$$(3.4) \quad \text{gr}\mathcal{M}_1 \rightarrow \text{gr}\mathcal{M}_2 \rightarrow \text{gr}\mathcal{M}_3.$$

By [3, Prop.A.17], if (3.4) is exact, then (3.3) is filtered exact.

Lemma 3.2. *Hypothesis (3.1) implies hypothesis (3.2).*

Proof. For $1 \leq d < r$ consider the assertion:

$$(3.5) \quad v_{d+1} \text{ acting on } \mathcal{D}_X / \sum_{j=1}^d \mathcal{D}_X \cdot v_j \text{ is injective}$$

and the assertion

$$(3.6) \quad \sigma(v_{d+1}) \text{ acting on } \mathcal{O}_{T^*X} / \sum_{j=1}^d \mathcal{O}_{T^*X} \cdot \sigma(v_j) \text{ is injective.}$$

It follows from the results mentioned above that (3.6) implies (3.5). Q.E.D.

We do not know if hypotheses (3.1) and (3.2) are equivalent.

Proposition 3.3. *Let \mathcal{I} satisfying (2.1) and (3.1) and let $\mathcal{M}_{\mathcal{I}}$ be the associated coherent \mathcal{D}_X -module. Then*

- (i) (v_1, \dots, v_r) is an involutive system of generators of $\widetilde{\mathcal{I}}$,
- (ii) $\text{char}(\mathcal{M}_{\mathcal{I}}) = \{(x; \xi) \in T^*X; v(x; \xi) = 0 \text{ for all } v \in \mathcal{I}\}$,
- (iii) $r = \text{rk}(\mathcal{I}) = \text{codim char}(\mathcal{M}_{\mathcal{I}})$,
- (iv) the projective dimension of $\mathcal{M}_{\mathcal{I}}$ is equal to $\text{rk}(\mathcal{I})$. In other words, $\mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}_{\mathcal{I}}, \mathcal{D}_X) = 0$ for $j \neq \text{rk}(\mathcal{I})$.

Note that the assertion on $\text{char}(\mathcal{M})$ was already obtained in [8, Cor. 4.13].

Proof. (i) The sequence of symbols being regular, the variety $\bigcap_{j=1}^r \sigma^{-1}(v_j)(0)$ has codimension r . Then apply [3, Prop. 2.12].

(ii) follows from (i).

(iii) It follows from (i) and (ii) that the variety $\text{char}(\mathcal{M})$ has codimension r . Hence, the leaves of this involutive manifold have dimension r and this dimension is the rank of \mathcal{I} .

(iv) Consider the Koszul complex $K^\bullet(\mathcal{D}_X, \{v_1, \dots, v_r\})$. The cohomology of this complex is concentrated in degree r by (3.2) (which follows from Lemma 3.2) and is isomorphic to $\mathcal{M}_{\mathcal{I}}$. Therefore $\mathcal{M}_{\mathcal{I}}$ admits a projective resolution of length $\leq r$ and the projective dimension of $\mathcal{M}_{\mathcal{I}}$ is $\leq r$. On the other hand, $\mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}_{\mathcal{I}}, \mathcal{D}_X) = 0$ for $j > \text{rk}(\mathcal{I})$ by [3, Th. 2.19]. Q.E.D.

Definition 3.4. (a) We set $\mathcal{D}_{\mathcal{J}} := \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_{\mathcal{J}}, \mathcal{M}_{\mathcal{J}})$ and $\mathcal{D}_{\mathcal{J}}^0 := H^0(\mathcal{D}_{\mathcal{J}})$.

(b) We denote by $\mathcal{D}\text{-irr}(\mathcal{J})$ the increasing sequence consisting of the integers k such that $H^k(\mathcal{D}_{\mathcal{J}}) \neq 0$ and we set $\mathrm{D}\text{-irr}(\mathcal{J}) = \sup(\mathcal{D}\text{-irr}(\mathcal{J}))$.

Remark 3.5. (i) The sequence $\mathcal{D}\text{-irr}(\mathcal{J})$ and the integer $\mathrm{D}\text{-irr}(\mathcal{J})$ are invariants of the foliation.

(ii) Note that $\mathcal{D}_{\mathcal{J}}^0$ is a ring and the restriction of $\mathcal{D}_{\mathcal{J}}$ to X_0 is concentrated in degree 0.

(iii) A similar ring to $\mathcal{D}_{\mathcal{J}}^0$ has also been constructed “at hands” in the real case for smooth foliations in [7].

(iv) The sheaf $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_{\mathcal{J}}, \mathcal{O}_X)$ is the sheaf of holomorphic functions that are constant along the leaves of \mathcal{J} .

Theorem 3.6. Assume (2.1) and (3.1). Then $\mathrm{D}\text{-irr}(\mathcal{J}) = \mathrm{irr}(\mathcal{J})$.

Proof. (A) First we prove the inequality $\mathrm{D}\text{-irr}(\mathcal{J}) \leq \mathrm{irr}(\mathcal{J})$.

(A)–(i) Assume first that $\mathrm{cork}(\mathcal{J}) = 0$, that is, $\mathrm{irr}(\mathcal{J}) = \mathrm{rk}(\mathcal{J})$. In this case the result follows from Proposition 3.3. Indeed, $\mathcal{E}xt_{\mathcal{D}_X}^k(\mathcal{M}_{\mathcal{J}}, \mathcal{N}) \simeq 0$ for all $k > \mathrm{rk}(\mathcal{J})$ and all \mathcal{D}_X -module \mathcal{N} .

(A)–(ii) Assume $\mathrm{cork}(\mathcal{J}) > 0$. Let (v_1, \dots, v_r) be as in Hypothesis 3.2. We may choose a local coordinate system (x_1, \dots, x_n) on X such that $v_1 = \partial_1$. Since $[v_1, v_j] = 0$, the v_j ’s do not depend on x_1 and of course, we may also assume that they do not depend on ∂_1 . (If $v_j = w_j + a(x)\partial_1$, we may replace v_j with w_j keeping the same hypotheses.)

(A)–(iii) Arguing by induction, we may assume that

$$(v_1, \dots, v_r) = (\partial_1, \dots, \partial_s, v_{s+1}, \dots, v_r)$$

where $s = \mathrm{cork}(\mathcal{J})$ and the v_j ’s ($j = s+1, \dots, r$) depend neither on (x_1, \dots, x_s) nor on $(\partial_1, \dots, \partial_s)$.

Let $X = X_1 \times X_2$ where $X_1 = \mathbb{C}^s$ and $X_2 = \mathbb{C}^{n-s}$. Let \mathcal{I}_j denote the ideal of \mathcal{D}_{X_j} generated by $(\partial_1, \dots, \partial_s)$ in case $j = 1$ and by (v_{s+1}, \dots, v_r) in case $j = 2$. Set

$$\widetilde{\mathcal{I}}_j := \mathcal{D}_{X_j} \cdot \mathcal{I}_j, \quad \mathcal{M}_j = \mathcal{D}_{X_j} / \widetilde{\mathcal{I}}_j \quad (j = 1, 2).$$

Note that $\mathcal{M}_1 \simeq \mathcal{O}_{X_1}$, the de Rham complex on X_1 . Then

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_{\mathcal{J}}, \mathcal{M}_{\mathcal{J}}) \simeq \mathrm{R}\mathcal{H}om_{\mathcal{D}_{X_2}}(\mathcal{M}_2, \mathrm{R}\mathcal{H}om_{\mathcal{D}_{X_1}}(\mathcal{M}_1, \mathcal{M}_{\mathcal{J}})).$$

Here we write for short \mathcal{D}_{X_j} instead of $p_j^{-1}\mathcal{D}_{X_j}$ where $p_j: X \rightarrow X_j$ is the projection, and similarly with the \mathcal{M}_j 's. An easy calculation gives

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_{X_1}}(\mathcal{M}_1, \mathcal{M}_{\mathcal{J}}) \simeq \mathcal{M}_2.$$

Hence, we are reduced to treat \mathcal{M}_2 in which case $\mathrm{irr}(\mathcal{I}_2) = \mathrm{rk}(\mathcal{I}_2)$. This completes the proof of (A) since $\mathrm{irr}(\mathcal{I}_2) = \mathrm{irr}(\mathcal{I})$.

(B) Let us prove that $H^r(\mathcal{D}_{\mathcal{J}}) \neq 0$. By the same argument as in (A) we may assume that $\mathrm{cork}(\mathcal{I}) = 0$. Since $\{v_1, \dots, v_r\}$ is a regular sequence, an easy calculation gives

$$H^r(\mathcal{D}_{\mathcal{J}}) \simeq \mathcal{D}_X / \sum_{j=1}^r (v_j \cdot \mathcal{D}_X + \mathcal{D}_X \cdot v_j).$$

We may assume that $X = \mathbb{C}^n$ and all v_j 's vanish at 0. Hence, we are reduced to prove that the equation

$$(3.7) \quad 1 = \sum_{j=1}^r (v_j \cdot A_j + B_j \cdot v_j)$$

has no solutions $A_j, B_j \in \mathcal{D}_X$. Let us argue by contradiction and apply the right-hand side of (3.7) to the holomorphic function 1. Since $v_j(1) = 0$, we get:

$$\left(\sum_{j=1}^r (v_j \cdot A_j + B_j \cdot v_j) \right)(1) = \sum_{j=1}^r (v_j \cdot A_j)(1) = \sum_{j=1}^r (v_j \cdot A_j)(1).$$

Since v_j vanishes at 0, the differential operator $v_j \cdot A_j$ also vanishes at 0 (meaning that, if one chooses a local coordinate system, all coefficients of this operator will vanish at 0). Therefore, $(v_j \cdot A_j)(1) = 0$. Q.E.D.

We examine some examples in which we calculate $H^i(\mathcal{D}_{\mathcal{J}})$ for $1 \leq i \leq r$.

Example 3.7. Let us particularize Theorem 3.6 in a simple situation.

Let $X = \mathbb{C}^2$ endowed with coordinates (x, y) . Consider the vector field $v = x\partial_x + y\partial_y$ and let \mathcal{J} be the Lie subalgebra of Θ_X generated by this vector field. If f is a section of \mathcal{O}_X , then $v(f) = 0$ implies that f is homogeneous of degree 0.

We have $X = X_0 \sqcup X_1$ where $X_1 = \{0\}$ and \mathcal{J} has rank 1 on X_0 , 0 on X_1 . The leaves of X_0 are the complex curves $\{(x, y); x^2 + y^2 = c\}$ ($c \in \mathbb{C}, c \neq 0$).

As already mentioned, the restriction of $\mathcal{D}_{\mathcal{J}}$ to X_0 is concentrated in degree 0. Let us calculate $H^1(\mathcal{D}_{\mathcal{J}})$. The module $\mathcal{M}_{\mathcal{J}} = \mathcal{D}_X / \mathcal{D}_X \cdot v$ is represented by the complex

$$0 \rightarrow \mathcal{D}_X^{-1} \xrightarrow{v} \mathcal{D}_X^0 \rightarrow 0$$

in which $\mathcal{D}_X^i = \mathcal{D}_X$ ($i = 0, -1$), v operates on the right and \mathcal{D}_X^0 is in degree 0. Therefore, $\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_{\mathcal{J}}, \mathcal{D}_X)$ is represented by the same complex where now v acts on the left and \mathcal{D}_X^0 is in degree 1. It follows that

$$H^1(\mathcal{D}_{\mathcal{J}}) \simeq \mathcal{D}_X / (v \cdot \mathcal{D}_X + \mathcal{D}_X \cdot v).$$

Since $1 \notin v \cdot \mathcal{D}_X + \mathcal{D}_X \cdot v$, we deduce that $H^1(\mathcal{D}_{\mathcal{J}}) \neq 0$.

Example 3.8. Assume that

$$(3.8) \begin{cases} \text{(i) } \mathcal{J} \text{ is of rank 2,} \\ \text{(ii) } \mathcal{J} \text{ is a Lie abelian subalgebra of } \Theta_X, \\ \text{(iii) } \mathrm{cork}(\mathcal{J}) = 0. \end{cases}$$

By [8, Rem. 4.3] and hypothesis (ii) there is a generator system $\{v_1, v_2\}$ of \mathcal{J} satisfying hypothesis (3.1). Therefore, Theorem 3.6 implies $H^2(\mathcal{D}_{\mathcal{J}}) \neq 0$. By hypothesis (iii) $\{v_1, v_2\}$ vanish at 0 hence $1 \notin v_1 \cdot \mathcal{D}_X + \mathcal{D}_X \cdot v_1$ and we have also $H^1(\mathcal{D}_{\mathcal{J}}) \neq 0$.

Some questions

- Is it possible to weaken Hypothesis 3.1?
- What is the geometric meaning of the sequence $\mathcal{D}\text{-irr}(\mathcal{J})$?

References

- [1] C. Camacho and P. Sad, *Invariant varieties through singularities of vector fields*, Ann. of Math. bf 115 (1982) pp. 579-595 (1982).
- [2] P. Etingof, T. Schedler and I. Losev, *Poisson traces and D-modules on Poisson varieties*, arXiv:0908.3868

- [3] M. Kashiwara, *D-modules and Microlocal Calculus*, Translations of Mathematical Monographs, **217** American Math. Soc. (2003).
- [4] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, Grundlehren der Math. Wiss. **292** Springer-Verlag (1990).
- [5] A. S. de Meideros, *Singular foliations and differential forms*, Ann. Fac. Sci. Toulouse, **9** pp. 451-466 (2000).
- [6] T. Nagano, *Linear differential systems with singularities and an application to transitive Lie algebras*, J. Math. Soc. Japan, **18** pp. 398–404 (1966).
- [7] N. Poncin, F. Radoux and R. Wolak, *Equivariant quantization of orbifolds*, [arXiv:1001.464](#)
- [8] T. Suwa, *\mathcal{D} -modules associated to complex analytic singular foliations*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **37** (1990) pp. 297-320.
- [9] F. Trèves, *Intrinsic stratifications of analytic varieties*, [arXiv:1402.0179](#)